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The Space Problem of the Calculus of Variations in Terms of Angle.

BY PAUL R. RIDER.

Introduction.

In her Göttingen thesis * Gernet treated the problem of the calculus of variations for three dimensions, considering an integral of the form

$$I=\int f\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}\right)dx.$$

Bliss and Mason † have developed the theory for the integral

$$I=\int f(x, y, z, x_t, y_t, z_t)\,dt,$$

where the arguments of f are functions of a parameter t.

It is the object of this paper to develop the theory for an integral in a different form from either of these, and to use the results in considering certain generalized definitions of angle and of solid angle. The first six sections treat the general theory. In § 1 the Euler equations are derived and the form of the equations of the extremals is shown. By means of the family of extremals through a given point the Jacobi necessary condition is derived in § 2. The Hilbert invariant integral and the Weierstrass E-function are taken up in § 3, and the necessary conditions of Weierstrass and Legendre are given in this section and in § 4. As the F_1-function, which first appears in § 1, is a definite expression instead of a factor of proportionality, the Legendre condition is more easily developed and is simpler than in the Weierstrass form of the problem in three-dimensional space as developed by Bliss and Mason. However, sufficient conditions are obtained for a weak extremum only—§ 5. The so-called transversality condition that must hold when one end-point is allowed to vary along a curve is given in § 6. In § 7 a transversal surface is studied and two generalized definitions of angle are made. Incidentally the differential equations of geodesics on the transversal surface are obtained· Finally, § 8 deals with a generalization of the notion of solid angle.

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* "Untersuchungen zur Variationsrechnung," Dissertation, Göttingen (1902).

† "The Properties of Curves in Space Which Minimize a Definite Integral," *Transactions of the American Mathematical Society*, Vol. IX (1908), p. 440.

§ 1. *The Euler Equations.*

Suppose that there is a curve defined by the functions

$$x=\phi(t), \quad y=\psi(t), \quad z=\chi(t). \tag{C}$$

The integral to be studied is

$$I=\int f(x, y, z, \tau, \sigma)\sqrt{x_t^2+y_t^2+z_t^2}\,dt, \tag{1}$$

where τ and σ are defined by the equations

$$\left.\begin{aligned} \frac{x_t}{\sqrt{x_t^2+y_t^2+z_t^2}} &= \cos\sigma\cos\tau, \\ \frac{y_t}{\sqrt{x_t^2+y_t^2+z_t^2}} &= \cos\sigma\sin\tau, \\ \frac{z_t}{\sqrt{x_t^2+y_t^2+z_t^2}} &= \sin\sigma. \end{aligned}\right\} \tag{2}$$

The geometric significance of τ and σ is evident from equations (2). If at any point on the curve C the positive tangent to the curve is constructed, then σ is the angle that this tangent makes with its own projection in the xy-plane, and τ is the angle that this projection makes with the x-axis.* It is assumed that the function $f(x, y, z, \tau, \sigma)$ is of class C^{iv} † with respect to each of its five arguments in a region $R(x, y, z, \tau, \sigma)$, and throughout the discussion the variables will be confined to this region. The curves considered will be represented in the parametric form (C), where ϕ, ψ, χ are functions of class C^{iv} defining values (x, y, z, τ, σ) interior to the region R for all values of t in the interval $t_0 \leqq t \leqq t_1$. Since τ and σ are to be definitely determinate for every value of t in this interval, it is assumed that x_t, y_t, z_t do not all vanish simultaneously, *i. e.*, that the curves under consideration are without singular points.

If the integral I is taken along a curve of the family

$$x=\phi(t, a), \quad y=\psi(t, a), \quad z=\chi(t, a), \tag{C_a}$$

where $\phi(t, a)$, $\psi(t, a)$, $\chi(t, a)$ are of class C^{iv} in the region

$$t_0 \leqq t \leqq t_1, \quad |a| \leqq k > 0,$$

and for $a=0$ define the curve C, then the value of I becomes a function $I(a)$, which for $a=0$ reduces to the value of I taken along C. Hence the derivative of I with respect to a must vanish for $a=0$.

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* Bliss has developed the theory for the integral $\int f(x, y, \tau)\sqrt{x_t^2+y_t^2}\,dt$. See "A New Form of the Simplest Problem of the Calculus of Variations," *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 405.

† For definitions of terms and notations used in this paper see Bolza, "Vorlesungen über Variationsrechnung."

This derivative is found to be

$$\frac{dI}{da}=\int_{t_0}^{t_1}\Big(f_x x_a+f_y y_a+f_z z_a+f_\tau \tau_a+f_\sigma \sigma_a \\ +f\frac{x_t x_{ta}+y_t y_{ta}+z_t z_{ta}}{x_t^2+y_t^2+z_t^2}\Big)\sqrt{x_t^2+y_t^2+z_t^2}dt. \qquad (3)$$

If the values of τ_a, σ_a are obtained from (2) and substituted in (3), after some reductions

$$\frac{dI}{da}=\int_{s_0}^{s_1}(f_x x_a+f_y y_a+f_z z_a+p x_{sa}+q y_{sa}+r z_{sa})\,ds, \qquad (4)$$

where

$$\left.\begin{aligned} p&=f\cos\sigma\cos\tau-f_\tau\frac{\sin\tau}{\cos\sigma}-f_\sigma\sin\sigma\cos\tau,\\ q&=f\cos\sigma\sin\tau+f_\tau\frac{\cos\tau}{\cos\sigma}-f_\sigma\sin\sigma\sin\tau,\\ r&=f\sin\sigma+f_\sigma\cos\sigma, \end{aligned}\right\} \qquad (5)$$

and where t has been replaced by s, the Euclidean length of arc.

The usual integration by parts gives

$$\frac{dI}{da}=\int_{s_0}^{s_1}(Px_a+Qy_a+Rz_a)\,ds+[px_a+qy_a+rz_a]_{s_0}^{s_1}, \qquad (6)$$

in which

$$P=f_x-\frac{dp}{ds},\quad Q=f_y-\frac{dq}{ds},\quad R=f_z-\frac{dr}{dz}. \qquad (7)$$

It follows at once that *the Euler equations are*

$$P=0,\quad Q=0,\quad R=0. \qquad (8)$$

These equations are not independent, however, for the relation

$$P\cos\sigma\cos\tau+Q\cos\sigma\sin\tau+R\sin\sigma\equiv 0 \qquad (9)$$

is readily verified.

When the differentiation indicated in (7) is carried out, equations (8) become

$$a_i\frac{d\tau}{ds}+b_i\frac{d\sigma}{ds}+c_i=0,\qquad i=1,\,2,\,3, \qquad (10)$$

where the *a's*, *b's* and *c's* are defined as follows:

$$a_1=f\cos\sigma\sin\tau+f_\tau\sin\sigma\tan\sigma\cos\tau-f_\sigma\sin\sigma\sin\tau+f_{\tau\tau}\frac{\sin\tau}{\cos\sigma}+f_{\tau\sigma}\sin\sigma\cos\tau,$$

$$b_1=f\sin\sigma\cos\tau+f_\tau\frac{\sin\sigma\sin\tau}{\cos^2\sigma}+f_{\tau\sigma}\frac{\sin\tau}{\cos\sigma}+f_{\sigma\sigma}\sin\sigma\cos\tau,$$

$$\begin{aligned} c_1=f_x(1-\cos^2\sigma\cos^2\tau)&-f_y\cos^2\sigma\sin\tau\cos\tau-f_z\sin\sigma\cos\sigma\cos\tau\\ &+f_{x\tau}\sin\tau\cos\tau+f_{y\tau}\sin^2\tau+f_{z\tau}\tan\sigma\sin\tau\\ &+f_{x\sigma}\sin\sigma\cos\sigma+f_{y\sigma}\sin\sigma\cos\sigma\sin\tau\cos\tau+f_{z\sigma}\sin^2\sigma\cos\tau, \end{aligned}$$

$$a_2 = -f\cos\sigma\cos\tau + f_\tau\sin\sigma\tan\sigma\sin\tau + f_\sigma\sin\sigma\cos\tau - f_{\tau\tau}\frac{\cos\tau}{\cos\sigma} + f_{\tau\sigma}\sin\sigma\sin\tau,$$

$$b_2 = f\sin\sigma\sin\tau - f_\tau\frac{\sin\sigma\cos\tau}{\cos^2\sigma} - f_{\tau\sigma}\frac{\cos\tau}{\cos\sigma} + f_{\sigma\sigma}\sin\sigma\sin\tau,$$

$$\begin{aligned} c_2 = {} & -f_x\cos^2\sigma\sin\tau\cos\tau + f_y(1-\cos^2\sigma\sin^2\tau) - f_z\sin\sigma\cos\sigma\sin\tau \\ & - f_{x\tau}\cos^2\tau - f_{y\tau}\sin\tau\cos\tau - f_{z\tau}\tan\sigma\cos\tau \\ & + f_{x\sigma}\sin\sigma\cos\sigma\sin\tau\cos\tau + f_{y\sigma}\sin\sigma\cos\sigma\sin^2\tau + f_{z\sigma}\sin^2\sigma\sin\tau, \end{aligned}$$

$$a_3 = -f_\tau\sin\sigma - f_{\tau\sigma}\cos\sigma,$$

$$b_3 = -f\cos\sigma - f_{\sigma\sigma}\cos\sigma,$$

$$\begin{aligned} c_3 = {} & -f_x\sin\sigma\cos\sigma\cos\tau - f_y\sin\sigma\cos\sigma\sin\tau + f_z\cos^2\sigma \\ & - f_{x\sigma}\cos^2\sigma\cos\tau - f_{y\sigma}\cos^2\sigma\sin\tau - f_{z\sigma}\sin\sigma\cos\sigma. \end{aligned}$$

The three equations (10) are linear and non-homogeneous in $\frac{d\tau}{ds}$, $\frac{d\sigma}{ds}$, and are consistent because of the identity (9). If any two-rowed determinant of the matrix

$$\left\| \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix} \right\| \tag{11}$$

is different from zero, equations (10) can be solved for $\frac{d\tau}{ds}$, $\frac{d\sigma}{ds}$. The values of these determinants are found to be

$$\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = f_1\cos\tau, \quad \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} = f_1\sin\tau, \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = f_1\tan\sigma, \tag{12}$$

where

$$f_1 = (f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau})(f + f_{\sigma\sigma}) - (f_\tau\tan\sigma + f_{\tau\sigma})^2. \tag{13}$$

If $f_1 \neq 0$, not all the determinants in (12) can be zero, and (10) can be solved, giving

$$\frac{d\tau}{ds} = A(x, y, z, \tau, \sigma), \quad \frac{d\sigma}{ds} = B(x, y, z, \tau, \sigma). \tag{14}$$

From (2),

$$\frac{dx}{ds} = \cos\sigma\cos\tau, \quad \frac{dy}{ds} = \cos\sigma\sin\tau, \quad \frac{dz}{ds} = \sin\sigma. \tag{15}$$

By reason of known existence theorems for differential equations, it may be stated that *in the region R, through the point x_0, y_0, z_0, and in the direction defined by τ_0, σ_0, one and only one extremal can be drawn. The equations of these extremals are found by integrating* (14) *and* (15). *They have the form*

$$\left. \begin{aligned} x &= \phi(s, x_0, y_0, z_0, \tau_0, \sigma_0), \\ y &= \psi(s, x_0, y_0, z_0, \tau_0, \sigma_0), \\ z &= \chi(s, x_0, y_0, z_0, \tau_0, \sigma_0). \end{aligned} \right\} \tag{16}$$

It may further be stated that the following initial conditions are satisfied:

$$\left.\begin{aligned}\phi(0, x_0, y_0, z_0, \tau_0, \sigma_0) &= x_0, & \phi_s(0, x_0, y_0, z_0, \tau_0, \sigma_0) &= \cos\sigma_0 \cos\tau_0,\\ \psi(0, x_0, y_0, z_0, \tau_0, \sigma_0) &= y_0, & \psi_s(0, x_0, y_0, z_0, \tau_0, \sigma_0) &= \cos\sigma_0 \sin\tau_0,\\ \chi(0, x_0, y_0, z_0, \tau_0, \sigma_0) &= z_0, & \chi_s(0, x_0, y_0, z_0, \tau_0, \sigma_0) &= \sin\sigma_0.\end{aligned}\right\} \quad (17)$$

§ 2. *The Jacobi Necessary Condition.*

If in equations (16) the constants x_0, y_0, z_0 are made equal to the coordinates of the point 0, and the constants τ_0, σ_0 are replaced by functions $\tau(u, v)$, $\sigma(u, v)$, which for $u=u_0$, $v=v_0$ reduce to the angles defining the direction of C at the point 0, these equations take the form

$$x=\phi(s, u, v), \quad y=\psi(s, u, v), \quad z=\chi(s, u, v). \qquad (18)$$

Equations (18) represent the extremal C_{01} (*i. e.*, the extremal from the point 0 to the point 1) for $u=u_0$, $v=v_0$ and satisfy the identities

$$x_0=\phi(0, u, v), \quad y_0=\psi(0, u, v), \quad z_0=\chi(0, u, v), \qquad (19)$$

which express the fact that all of the extremals pass through the point 0 for $s=0$.

The functional determinant of equations (18) is

$$\Delta(s, u, v) \equiv \begin{vmatrix} \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{vmatrix},$$

and it can be shown in the usual way that if an extremal C_{01} makes the integral I either a maximum or a minimum, it can not have on it a point 2 at which the determinant Δ vanishes, *i. e.*, a point conjugate to the initial point 0, provided that at least one of the three-rowed determinants of the matrix

$$\begin{Vmatrix} \Delta_s & \Delta_u & \Delta_v \\ \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{Vmatrix}$$

is different from zero at the point 2.

§ 3. *The Hilbert Invariant Integral, the Weierstrass E-function, and the Necessary Condition of Weierstrass.*

An extremal arc C_{01} that does not have on it a point 2 conjugate to the initial point 0, can be imbedded in a two-parameter family of extremals through a point 0′ in the neighborhood of 0 and in the order 0′01, whose determinant Δ is different from zero at every point of C_{01}. The family of extremals thus forms a field about the extremal C, which is given by the parameter values $u=u_0$, $v=v_0$. Through each point of the region

$$|u-u_0| \leqq \delta, \quad |v-v_0| \leqq \delta, \tag{20}$$

provided δ, which is a positive constant, is sufficiently small, there passes one, and but one, extremal of the set. Furthermore, δ can be so restricted that in the region (20) Δ is different from zero, and the equations

$$x=\phi(s, u, v), \quad y=\psi(s, u, v), \quad z=\chi(s, u, v) \tag{21}$$

have unique solutions

$$s=s(x, y, z), \quad u=u(x, y, z), \quad v=v(x, y, z) \tag{22}$$

of class C'. Consider a curve

$$x=\bar{x}(a), \quad y=\bar{y}(a), \quad z=\bar{z}(a) \tag{$\bar{C}$}$$

that lies entirely within the region (20). Equations (22) and ($\bar{C}$) determine u and v as functions of a. Substitution of these functions in (21) gives a one-parameter family of extremals

$$x=\phi(s, a), \quad y=\psi(s, a), \quad z=\chi(s, a), \tag{23}$$

all of which are of course members of the two-parameter family (21).

The value of the integral I taken along an extremal of the set (23) from the point 0, where $s=0$ to the point 4, where the extremal intersects the curve $\bar{C}$ is obviously a function of a. By the method used in deriving the Euler equations it is found that

$$\frac{d}{da} I(C_{0'4}) = f \frac{\partial s}{\partial a}\Big|^4 + [p\phi_a + q\psi_a + r\chi_a]_{0'}^4, \tag{24}$$

p, q, r being defined by (5). The terms in the brackets all vanish at the point $0'$, for all of the extremals pass through that point, and consequently ϕ_a, ψ_a, χ_a are zero there. At the point 4,

$$\phi_s \frac{\partial s}{\partial a} + \phi_a = \bar{x}_a, \quad \psi_s \frac{\partial s}{\partial a} + \psi_a = \bar{y}_a, \quad \chi_s \frac{\partial s}{\partial a} + \chi_a = \bar{z}_a,$$

and, furthermore, ϕ, ψ, χ, τ, σ of (23) are equal, respectively, to ϕ, ψ, χ, τ, σ of (21) where the arguments s, u, v of the latter are expressed as functions of a by means of equations (22) and ($\bar{C}$). Since

$$p\phi_s + q\psi_s + r\chi_s = f,$$

equation (24) may be written

$$\frac{d}{da} I(C_{0'4}) = [p\bar{x}_a + q\bar{y}_a + r\bar{z}_a]\,\Big|^4,$$

the arguments $\phi(s, a)$, $\psi(s, a)$, $\chi(s, a)$, $\tau(s, a)$, $\sigma(s, a)$ of f, f_τ, f_σ in (24) having been replaced by $\phi(s, u, v)$, $\psi(s, u, v)$, $\chi(s, u, v)$, $\tau(s, u, v)$, $\sigma(s, u, v)$ considered as functions of a.

Let a_4 be the value of a that gives the extremal $C_{0'4}$ and a_5 the value of a that gives the extremal $C_{0'5}$, where 5 as well as 4 is a point of the curve $\overline{C}$. Then

$$\begin{aligned}
I(C_{0'5})-I(C_{0'4}) &= \int_{a_4}^{a_5}(p\bar{x}_a+q\bar{y}_a+r\bar{z}_a)\,da \\
&= \int_{a_4}^{a_5}\Big[\Big(f\cos\sigma\cos\tau-f_\tau\frac{\sin\tau}{\cos\sigma}-f_\sigma\sin\sigma\cos\tau\Big)\cos\bar{\sigma}\cos\bar{\tau} \\
&\qquad +\Big(f\cos\sigma\sin\tau+f_\tau\frac{\cos\tau}{\cos\sigma}-f_\sigma\sin\sigma\sin\tau\Big)\cos\bar{\sigma}\sin\bar{\tau} \\
&\qquad +(f\sin\sigma+f_\sigma\cos\sigma)\sin\bar{\sigma}\Big]\sqrt{\bar{x}_a^2+\bar{y}_a^2+\bar{z}_a^2}\,da \\
&= \int_{a_4}^{a_5}\Big\{f[\cos\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau)+\sin\sigma\sin\bar{\sigma}]+f_\tau\frac{\cos\bar{\sigma}}{\cos\sigma}\sin(\bar{\tau}-\tau) \\
&\qquad -f_\sigma[\sin\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau)-\cos\sigma\sin\bar{\sigma}]\Big\}\sqrt{\bar{x}_a^2+\bar{y}_a^2+\bar{z}_a^2}\,da,
\end{aligned}$$

where $\bar{\tau}$, $\bar{\sigma}$ are the angles defining the direction of the curve $\overline{C}$.

Since $I(C_{0'5})-I(C_{0'4})$ *does not depend on the form of the curve* $\overline{C}$, *but merely on the points* 4 *and* 5, *it is seen that the integral*

$$\begin{aligned}
I^* = \int\Big\{&f[\cos\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau)+\sin\sigma\sin\bar{\sigma}]+f_\tau\frac{\cos\bar{\sigma}}{\cos\sigma}\sin(\bar{\tau}-\tau) \\
&-f_\sigma[\sin\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau)-\cos\sigma\sin\bar{\sigma}]\Big\}\sqrt{\bar{x}_a^2+\bar{y}_a^2+\bar{z}_a^2}\,da,
\end{aligned}$$

taken along a curve $\overline{C}$ *in the field of extremals does not depend on the path of integration, but is a function of the end points only.* It is Hilbert's invariant integral and along an extremal reduces to the integral I, as is seen at once if $\bar{\tau}$, $\bar{\sigma}$ are set equal to τ, σ, respectively.

The Weierstrass E-function is the function

$$\begin{aligned}
e(x, y, z;\ \tau, \sigma;\ \bar{\tau}, \bar{\sigma}) &= f(x, y, z, \bar{\tau}, \bar{\sigma}) \\
&\quad -f[\cos\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau)+\sin\sigma\sin\bar{\sigma}]-f_\tau\frac{\cos\bar{\sigma}}{\cos\sigma}\sin(\bar{\tau}-\tau) \\
&\quad +f_\sigma[\sin\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau)-\cos\sigma\sin\bar{\sigma}]
\end{aligned}$$

occurring in the integrand of the difference

$$I(\overline{C})-I^*(\overline{C})=\int e\sqrt{\bar{x}_a^2+\bar{y}_a^2+\bar{z}_a^2}\,da.$$

Weierstrass's necessary condition follows at once: *If the arc* C_{01} *is a solution of the Euler equations that minimizes the integral* I *with respect to* *admissible curves* $\bar{C}_{01}$ *lying entirely within the field of extremals surrounding* C_{01}, *then the condition*

$$e(x,\, y,\, z;\; \tau,\, \sigma;\; \bar\tau,\, \bar\sigma) \geqq 0$$

must be satisfied at every point of C_{01} *and for every direction* $\bar\tau$, $\bar\sigma$. *If* C_{01} *maximizes* I, e *must be* $\leqq 0$.

§ 4. *The Necessary Condition of Legendre.*

In order to deduce the necessary condition of Legendre, the connection between the e-function of the last section and the function f_1 defined in § 1 must be established. To do this, expand by Taylor's series with a remainder, each term of e considered as a function of $\bar\tau$ and $\bar\sigma$. It is found that

$$\begin{aligned} f(x,\, y,\, z,\, \bar\tau,\, \bar\sigma) &= f + (\bar\tau - \tau) f_\tau + (\bar\sigma - \sigma) f_\sigma \\ &\quad + \tfrac{1}{2}[(\bar\tau-\tau)^2 f_{\tau\tau} + 2(\bar\tau-\tau)(\bar\sigma-\sigma) f_{\tau\sigma} + (\bar\sigma-\sigma)^2 f_{\sigma\sigma}] + h_1, \\ f[\cos\sigma \cos\bar\sigma \cos(\bar\tau-\tau) + \sin\sigma \sin\bar\sigma] & \\ &= f + \tfrac{1}{2}[-(\bar\tau-\tau)^2 f \cos^2\sigma - (\bar\sigma-\sigma)^2 f] + h_2 f, \\ f_\tau \frac{\cos\bar\sigma}{\cos\sigma} \sin(\bar\tau-\tau) &= (\bar\tau-\tau) f_\tau - (\bar\tau-\tau)(\bar\sigma-\sigma) f_\tau \tan\sigma + h_3 f_\tau, \\ f_\sigma[\sin\sigma \cos\bar\sigma \cos(\bar\tau-\tau) - \cos\sigma \sin\bar\sigma] & \\ &= -(\bar\sigma-\sigma) f_\sigma - \tfrac{1}{2}(\bar\tau-\tau)^2 f_\sigma \sin\sigma \cos\sigma + h_4 f_\sigma . \end{aligned}$$

The remainder h_1 is homogeneous of degree 3 in $\bar\tau-\tau$, $\bar\sigma-\sigma$, and contains third partial derivatives of f with mean value arguments τ^*, σ^*. Since it has been assumed that f is of class C^{iv} with respect to all of its arguments, these partial derivatives are finite, and h_1 vanishes as $\bar\tau-\tau$, $\bar\sigma-\sigma$ both approach zero. The expressions h_2, h_3, h_4 are also homogeneous of degree 3 in $\bar\tau-\tau$, $\bar\sigma-\sigma$, and, consequently, converge to zero with these two quantities. Thus,

$$\begin{aligned} e(x,\, y,\, z;\; \tau,\, \sigma;\; \bar\tau,\, \bar\sigma) &= \tfrac{1}{2}(\bar\tau-\tau)^2 (f \cos^2\sigma - f_\sigma \sin\sigma \cos\sigma + f_{\tau\tau}) \\ &\quad + (\bar\tau-\tau)(\bar\sigma-\sigma)(f_\tau \tan\sigma + f_{\tau\sigma}) + \tfrac{1}{2}(\bar\sigma-\sigma)^2 (f + f_{\sigma\sigma}) + h, \end{aligned}$$

where h is a finite expression that is homogeneous of degree 3 in $\bar\tau-\tau$, $\bar\sigma-\sigma$.

Consider now the quadratic form

$$K \equiv \xi^2 (f \cos^2\sigma - f_\sigma \sin\sigma \cos\sigma + f_{\tau\tau}) + 2\xi\eta (f_\tau \tan\sigma + f_{\tau\sigma}) + \eta^2 (f + f_{\sigma\sigma})$$

in the real variables ξ, η. If the arc C_{01} furnishes a minimum (either strong or weak) for the integral I, then K must be $\geqq 0$ at all points of the arc C_{01} and for all values of ξ and η.

For if ξ, η were a pair of values making K negative at some point of C_{01}, $\bar\tau$ and $\bar\sigma$ could be chosen so that $\bar\tau-\tau=\varepsilon\xi$, $\bar\sigma-\sigma=\varepsilon\eta$, and for sufficiently small values of ε the e-function would be negative. But it was seen in the preceding section that e must be $\geqq 0$ for a minimum. Similarly, for a maximum, K must be $\leqq 0$.

A necessary and sufficient condition that K be definite is that

$$(f_\tau \tan\sigma + f_{\tau\sigma})^2 - (f\cos^2\sigma - f_\sigma \sin\sigma\cos\sigma + f_{\tau\tau})(f+f_{\sigma\sigma}) < 0,$$

i. e., that $f_1 > 0$. Therefore, for a so-called regular problem (one in which $f_1 \neq 0$), f_1 must be positive if the extremal C_{01} furnishes either a minimum or a maximum for the integral I.

The relations

$$\left.\begin{aligned} &(f+f_{\sigma\sigma})K = f_1\xi^2 + [(f_\tau\tan\sigma + f_{\tau\sigma})\xi + (f+f_{\sigma\sigma})\eta]^2,\\ &(f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau})K\\ &\quad = [(f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau})\xi + (f_\tau\tan\sigma + f_{\tau\sigma})\eta]^2 + f_1\eta^2 \end{aligned}\right\} \qquad (25)$$

are easily verified. If it is assumed that $f_1 > 0$, it follows that

$$K, \quad f+f_{\sigma\sigma}, \quad f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau}$$

must all have the same sign. But if f_1 is positive, neither $f+f_{\sigma\sigma}$ nor $f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau}$ can be equal to zero. Consequently, they must be greater than zero for a minimum and less than zero for a maximum.

§ 5. *Sufficient Conditions for a Weak Extremum.*

The e-function can be expressed in forms similar to (25) of the preceding section:

$$\begin{aligned} &2(f+f_{\sigma\sigma})e = f_1(\bar\tau-\tau)^2 + [(f_\tau\tan\sigma + f_{\tau\sigma})(\bar\tau-\tau) + (f+f_{\sigma\sigma})(\bar\sigma-\sigma)]^2 + 2(f+f_{\sigma\sigma})h\\ &2(f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau})e\\ &\quad = [(f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau})(\bar\tau-\tau) + (f_\tau\tan\sigma + f_{\tau\sigma})(\bar\sigma-\sigma)]^2\\ &\qquad\qquad + f_1(\bar\sigma-\sigma)^2 + 2(f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau})h \end{aligned}$$

where h, as previously stated, is homogeneous of degree 3 in $\bar\tau-\tau$, $\bar\sigma-\sigma$.

For a regular problem, $f+f_{\sigma\sigma}$ and $f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau}$ are either both positive or else both negative. Assume them positive, *i. e.*, assume that Legendre's necessary condition for a minimum is fulfilled. Then, at least if $|\bar\tau-\tau|$ and $|\bar\sigma-\sigma|$ are less than some positive constant γ, the e-function is positive except when $\bar\tau=\tau$, $\bar\sigma=\sigma$, in which case it vanishes.

It follows that if along the extremal arc C_{01} Jacobi's necessary condition for an extremum and Legendre's necessary condition for a minimum are satisfied, C_{01} will furnish at least a weak minimum for the integral I. A similar theorem of course holds for a weak maximum.

§ 6. *The Transversality Condition.*

It can easily be proved that if one end point of the extremal C is allowed to vary along a curve $\overline{C}$, then the condition

$$f[\cos\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau)+\sin\sigma\sin\bar{\sigma}]+f_\tau\frac{\cos\bar{\sigma}}{\cos\sigma}\sin(\bar{\tau}-\tau)$$
$$-f_\sigma[\sin\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau)-\cos\sigma\sin\bar{\sigma}]=0$$

must be satisfied at the point of intersection of C and $\overline{C}$.

When $f=1$ this condition reduces to the condition for perpendicularity as is to be expected.

When $\bar{\sigma}=\sigma=0$ it reduces to the condition as given by Bliss for the problem in two dimensions.

§ 7. *A Generalized Notion of Angle.*

Consider the two-parameter family of extremals (18) that pass through the point 0. On each of the extremals take an extremal arc such that the value of the integral I taken along the arc from the point 0 to the other end of the arc shall be equal to a constant ρ. The value of s determining the end-point of this arc is defined by the equation

$$\rho=\int_0^s f(\phi,\psi,\chi,\tau,\sigma)\sqrt{\phi_s^2+\psi_s^2+\chi_s^2}\,ds,\quad \phi_s^2+\psi_s^2+\chi_s^2=1. \tag{26}$$

The locus of the terminal points of these arcs is a so-called transversal surface. Since ρ may be considered as the generalized distance from the point 0 to this surface, the surface itself may be thought of as a generalized sphere whose center is the point 0 and whose radius is ρ.

Assuming that f is greater than zero, $\frac{\partial\rho}{\partial s}=f$ is different from zero at the point 0, and (26) can be solved for s in terms of ρ, τ_0, σ_0, and the solution will have continuous first derivatives with respect to all three arguments in a properly chosen region. Since

$$\frac{\partial s}{\partial\rho}=\frac{1}{f}>0$$

in the vicinity of the point 0, the value of s will increase from zero with ρ. The equations of the transversal surface are found by substituting in equations (16) the value of s obtained from (26). The parameters of the surface are τ_0 and σ_0. It is convenient to introduce here the following functions of the theory of surfaces:

$$E=x_{\tau_0}^2+y_{\tau_0}^2+z_{\tau_0}^2,\quad F=x_{\tau_0}x_{\sigma_0}+y_{\tau_0}y_{\sigma_0}+z_{\tau_0}z_{\sigma_0},\quad G=x_{\sigma_0}^2+y_{\sigma_0}^2+z_{\sigma_0}^2.$$

Consider any curve $\bar{C}$ on the transversal surface, and suppose that τ_0 and σ_0 are functions of a single parameter a. Then the value of the integral I, taken over this curve from the point $A_0(a=a_0)$ to the point $A_1(a=a_1)$ is

$$\lambda=\int_{a_0}^{a_1} f(\bar{x}, \bar{y}, \bar{z}, \bar{\tau}, \bar{\sigma}) \sqrt{E\tau_0'^2+2F\tau_0'\sigma_0'+G\sigma_0'^2}\,da, \tag{27}$$

where $\bar{x}, \bar{y}, \bar{z}, \bar{\tau}, \bar{\sigma}$ refer to $\bar{C}$, and where accents denote differentiation with respect to a. The quantity λ may be thought of as the generalized length of the curve $\bar{C}$ from the point A_0 to the point A_1.

Let the angle through which the extremal OA_1 has passed in moving to its position from the position OA_0 be defined as the limit, as ρ approaches zero, of the ratio $\dfrac{\lambda}{\rho}$.

The process of calculating this limit is somewhat lengthy, but can be accomplished in a perfectly straightforward manner. By partial differentiation with respect to τ_0 and σ_0, from (26) are obtained the two equations:

$$0=f(\phi, \psi, \chi, \tau, \sigma)\frac{\partial s}{\partial \tau_0}+\int_0^s \frac{\partial}{\partial \tau_0}[f(\phi, \psi, \chi, \tau, \sigma)\sqrt{\phi_s^2+\psi_s^2+\chi_s^2}]\,ds,$$

$$0=f(\phi, \psi, \chi, \tau, \sigma)\frac{\partial s}{\partial \sigma_0}+\int_0^s \frac{\partial}{\partial \sigma_0}[f(\phi, \psi, \chi, \tau, \sigma)\sqrt{\phi_s^2+\psi_s^2+\chi_s^2}]\,ds,$$

which reduce, when integrated by parts, to

$$\left.\begin{aligned} 0&=f\frac{\partial s}{\partial \tau_0}+p\phi_{\tau_0}+q\psi_{\tau_0}+r\chi_{\tau_0}, \\ 0&=f\frac{\partial s}{\partial \sigma_0}+p\phi_{\sigma_0}+q\psi_{\sigma_0}+r\chi_{\sigma_0}, \end{aligned}\right\} \tag{28}$$

Now

$$x_{\tau_0}=\phi_s\frac{\partial s}{\partial \tau_0}+\phi_{\tau_0}=\cos\sigma\cos\tau\frac{\partial s}{\partial \tau_0}+\phi_{\tau_0},$$

and there are similar expressions for y_{τ_0}, z_{τ_0}. By using the expression for $\dfrac{\partial s}{\partial \tau_0}$ obtained from (28), and making some reductions, the values of x_{τ_0}, y_{τ_0}, z_{τ_0} are found to be

$$\begin{aligned} x_{\tau_0}=\frac{1}{f}\,[&\phi_{\tau_0}(f-f\cos^2\sigma\cos^2\tau+f_\tau\sin\tau\cos\tau+f_\sigma\sin\sigma\cos\sigma\cos^2\tau) \\ &+\psi_{\tau_0}(-f\cos^2\sigma\sin\tau\cos\tau-f_\tau\cos^2\tau+f_\sigma\sin\sigma\cos\sigma\sin\tau\cos\tau) \\ &+\chi_{\tau_0}(-f\sin\sigma\cos\sigma\cos\tau-f_\sigma\cos^2\sigma\cos\tau)] \\ =\frac{1}{f}\,[&q(\phi_{\tau_0}\cos\sigma\sin\tau-\psi_{\tau_0}\cos\sigma\cos\tau)-r(\chi_{\tau_0}\cos\sigma\cos\tau-\phi_{\tau_0}\sin\sigma)], \end{aligned}$$

$$\begin{aligned} y_{\tau_0} = \frac{1}{f} [&\phi_{\tau_0}(-f \cos^2\sigma \sin^2\tau + f_\tau \sin^2\tau + f_\sigma \sin\sigma \cos\sigma \sin\tau \cos\tau) \\ &+ \psi_{\tau_0}(f - f\cos^2\sigma \sin^2\tau - f_\tau \sin\tau \cos\tau + f_\sigma \sin\sigma \cos\sigma \sin^2\tau) \\ &+ \chi_{\tau_0}(-f \sin\sigma \cos\sigma \sin\tau - f_\sigma \cos^2\sigma \sin\tau)] \\ = \frac{1}{f} [&r(\psi_{\tau_0} \sin\sigma - \chi_{\tau_0} \cos\sigma \sin\tau) - p(\phi_{\tau_0} \cos\sigma \sin\tau - \psi_{\tau_0} \cos\sigma \cos\tau)], \end{aligned}$$

$$\begin{aligned} z_{\tau_0} = \frac{1}{f} [&\phi_{\tau_0}(-f \sin\sigma \cos\sigma \cos\tau + f_\tau \tan\sigma \sin\tau + f_\sigma \sin^2\sigma \cos\tau) \\ &+ \psi_{\tau_0}(-f \sin\sigma \cos\sigma \sin\tau - f_\tau \tan\sigma \cos\tau + f_\sigma \sin^2\sigma \sin\tau) \\ &+ \chi_{\tau_0}(f - f \sin^2\sigma - f_\sigma \sin\sigma \cos\sigma)] \\ = \frac{1}{f} [&p(\chi_{\tau_0} \cos\sigma \cos\tau - \phi_{\tau_0} \sin\sigma) - q(\psi_{\tau_0} \sin\sigma - \chi_{\tau_0} \cos\sigma \sin\tau)]. \end{aligned}$$

The values of x_{σ_0}, y_{σ_0}, z_{σ_0} are obtained by replacing the subscript τ_0 by σ_0 in the right-hand members of the last three equations.

Some limits which are needed in finding the value of the limit of $\frac{\lambda}{\rho}$ are now to be calculated. Since

$$\frac{ds}{d\rho} = \frac{1}{f(\phi, \psi, \chi, \tau, \sigma)},$$

it follows that

$$\lim_{\rho=0} \frac{s}{\rho} = \frac{1}{f(x_0, y_0, z_0, \tau_0, \sigma_0)}. \tag{29}$$

By the law of the mean

$$\begin{aligned} \phi_{\tau_0}(s, \tau_0, \sigma_0) &= \phi_{\tau_0}(0, \tau_0, \sigma_0) + s\phi_{s\tau_0}(\theta s, \tau_0, \sigma_0), \quad 0 < \theta < 1, \\ &= s\phi_{s\tau_0}(\theta s, \tau_0, \sigma_0), \end{aligned}$$

since from (17) $\phi_{\tau_0}(0, \tau_0, \sigma_0)$ is identically zero. Thus,

$$\lim_{\rho=0} \frac{1}{\rho} \phi_{\tau_0}(s, \tau_0, \sigma_0) = \lim_{\rho=0} \frac{1}{\rho} s\phi_{s\tau_0}(\theta s, \tau_0, \sigma_0) = -\frac{\cos\sigma_0 \sin\tau_0}{f(x_0, y_0, z_0, \tau_0, \sigma_0)},$$

using (29) and (17). Similarly,

$$\lim_{\rho=0} \frac{1}{\rho} \psi_{\tau_0}(s, \tau_0, \sigma_0) = \frac{\cos\sigma_0 \cos\tau_0}{f(x_0, y_0, z_0, \tau_0, \sigma_0)},$$

$$\lim_{\rho=0} \frac{1}{\rho} \chi_{\tau_0}(s, \tau_0, \sigma_0) = 0,$$

$$\lim_{\rho=0} \frac{1}{\rho} \phi_{\sigma_0}(s, \tau_0, \sigma_0) = -\frac{\sin\sigma_0 \cos\tau_0}{f(x_0, y_0, z_0, \tau_0, \sigma_0)},$$

$$\lim_{\rho=0} \frac{1}{\rho} \psi_{\sigma_0}(s, \tau_0, \sigma_0) = -\frac{\sin\sigma_0 \sin\tau_0}{f(x_0, y_0, z_0, \tau_0, \sigma_0)},$$

$$\lim_{\rho=0} \frac{1}{\rho} \chi_{\sigma_0}(s, \tau_0, \sigma_0) = \frac{\cos\sigma_0}{f(x_0, y_0, z_0, \tau_0, \sigma_0)}.$$

From these limits may be computed

$$\lim_{\rho=0}\frac{1}{\rho}x_{\tau_0}=-\frac{1}{f^2}\,(q\cos^2\sigma_0+r\sin\sigma_0\cos\sigma_0\sin\tau_0)$$
$$=-\frac{1}{f^2}\cos\sigma_0(f\sin\tau_0+f_\tau\cos\tau_0),$$

$$\lim_{\rho=0}\frac{1}{\rho}y_{\tau_0}=\frac{1}{f^2}\,(p\cos^2\sigma_0+r\sin\sigma_0\cos\sigma_0\cos\tau_0)$$
$$=\frac{1}{f^2}\cos\sigma_0(f\cos\tau_0-f_\tau\sin\tau_0),$$

$$\lim_{\rho=0}\frac{1}{\rho}z_{\tau_0}=\frac{1}{f^2}\,(p\sin\sigma_0\cos\sigma_0\sin\tau_0-q\sin\sigma_0\cos\sigma_0\cos\tau_0)=-\frac{1}{f^2}f_\tau\sin\sigma_0,$$

$$\lim_{\rho=0}\frac{1}{\rho}x_{\sigma_0}=-\frac{1}{f^2}r\cos\tau_0=-\frac{1}{f^2}\cos\tau_0(f\sin\sigma_0+f_\sigma\cos\sigma_0),$$

$$\lim_{\rho=0}\frac{1}{\rho}y_{\sigma_0}=-\frac{1}{f^2}r\sin\tau_0=-\frac{1}{f^2}\sin\tau_0(f\sin\sigma_0+f_\sigma\cos\sigma_0),$$

$$\lim_{\rho=0}\frac{1}{\rho}z_{\sigma_0}=\frac{1}{f^2}\,(p\cos\tau_0+q\sin\tau_0)=\frac{1}{f^2}\,(f\cos\sigma_0-f_\sigma\sin\sigma_0),$$

from which it follows that

$$\lim_{\rho=0}\frac{E}{\rho^2}=\frac{1}{f^4}\,(f^2\cos^2\sigma_0+f_\tau^2),\quad \lim_{\rho=0}\frac{F}{\rho^2}=\frac{1}{f^4}f_\tau f_\sigma,\quad \lim_{\rho=0}\frac{G}{\rho^2}=\frac{1}{f^4}\,(f^2+f_\sigma^2).$$

Going back to (27), *it is found that*

$$\lim_{\rho=0}\frac{\lambda}{\rho}=\int_{a_0}^{a_1}\frac{f(x_0,y_0,z_0,\bar\tau_0,\bar\sigma_0)}{f(x_0,y_0,z_0,\tau_0,\sigma_0)}\sqrt{(f^2\cos^2\sigma_0+f_\tau^2)\tau_0'^2+2f_\tau f_\sigma\tau_0'\sigma_0'+(f^2+f_\sigma^2)\sigma_0'^2}\,da,\quad(30)$$

which is the generalized expression for angle.

In the case of Euclidean geometry, $f=1$, the extremals are straight lines, and the transversal surface is the sphere

$$x=\rho\cos\sigma_0\cos\tau_0,\quad y=\rho\cos\sigma_0\sin\tau_0,\quad z=\rho\sin\sigma_0,$$

for which $E=\rho^2\cos^2\sigma_0$, $F=0$, $G=\rho^2$. Since f_τ and f_σ both vanish, the expression (30) reduces to

$$\int_{a_0}^{a_1}\sqrt{\cos^2\sigma_0\tau_0'^2+\sigma_0'^2}\,da,\quad(31)$$

which is the ordinary definition of the angle through which the line OA has moved in passing from the position OA_0 to the position OA_1, the point A traversing any curve on the sphere.

A slightly less general definition of angle in Euclidean geometry is the value of (31) when the point A is restricted to move along a great circle of the sphere, *i. e.*, when the line moves from OA_0 to OA_1 keeping always in the same plane. It will be shown that a curve $\overline{C}$ on the transversal surface (or in fact on any surface) that minimizes the integral (27) with respect to neighboring curves on the surface, must satisfy Euler equations, which for $f=1$ reduce to the differential equations of geodesics. Since the geodesics of a sphere are great circles, the value of the integral in (30) taken over such a curve as $\overline{C}$, which may be called a generalized geodesic, obviously reduces for Euclidean geometry to the length of the circular arc A_0A_1 divided by the radius of the sphere, which is the definition of the plane angle that OA_1 makes with OA_0. For simplicity write

$$\lambda=\int_{a_0}^{a_1}\overline{f}\Gamma da,$$

where

$$\overline{f}=f(\overline{x},\overline{y},\overline{z},\overline{\tau},\overline{\sigma}),\quad \Gamma=\sqrt{E\tau_0'^2+2F\tau_0'\sigma_0'+G\sigma_0'^2}.$$

It is readily found that *the curve* $\overline{C}$ *mentioned above must satisfy the two equations*

$$\begin{aligned}\overline{x}_{\tau_0}&\left[\overline{f}_x\Gamma+\frac{d}{da}\left(\overline{f}_\tau\frac{\sin\overline{\tau}}{\cos\overline{\sigma}}+\overline{f}_\sigma\sin\overline{\sigma}\cos\overline{\tau}\right)\right]\\&+\overline{y}_{\tau_0}\left[\overline{f}_y\Gamma-\frac{d}{da}\left(\overline{f}_\tau\frac{\cos\overline{\tau}}{\cos\overline{\sigma}}-\overline{f}_\sigma\sin\overline{\sigma}\sin\overline{\tau}\right)\right]\\&+\overline{z}_{\tau_0}\left[\overline{f}_z\Gamma-\frac{d}{da}\overline{f}_\sigma\cos\overline{\sigma}\right]+\overline{f}\Gamma_{\tau_0}-\frac{d}{da}\overline{f}\Gamma_{\tau'_0}=0,\end{aligned}\tag{32}$$

$$\begin{aligned}\overline{x}_{\sigma_0}&\left[\overline{f}_x\Gamma+\frac{d}{da}\left(\overline{f}_\tau\frac{\sin\overline{\tau}}{\cos\overline{\sigma}}+\overline{f}_\sigma\sin\overline{\sigma}\cos\overline{\tau}\right)\right]\\&+\overline{y}_{\sigma_0}\left[\overline{f}_y\Gamma-\frac{d}{da}\left(\overline{f}_\tau\frac{\cos\overline{\tau}}{\cos\overline{\sigma}}-\overline{f}_\sigma\sin\overline{\sigma}\sin\overline{\tau}\right)\right]\\&+\overline{z}_{\sigma_0}\left[\overline{f}_z\Gamma-\frac{d}{da}\overline{f}_\sigma\cos\overline{\sigma}\right]+\overline{f}\Gamma_{\sigma_0}-\frac{d}{da}\overline{f}\Gamma_{\sigma'_0}=0,\end{aligned}\tag{33}$$

where $\overline{\tau}$ and $\overline{\sigma}$, the angles determining direction on the curve $\overline{C}$, are defined analytically by the equations

$$\cos\overline{\sigma}\cos\overline{\tau}=\frac{\overline{x}'}{\sqrt{\overline{x}'^2+\overline{y}'^2+\overline{z}'^2}}=\frac{\overline{x}'}{\Gamma},$$

$$\cos\overline{\sigma}\sin\overline{\tau}=\frac{\overline{y}'}{\Gamma},\quad \sin\overline{\sigma}=\frac{\overline{z}'}{\Gamma}.$$

When $f=1$ equations (32) and (33) reduce to

$$\Gamma_{\tau_0}-\frac{d}{da}\Gamma_{\tau'_0}=0,\quad \Gamma_{\sigma_0}-\frac{d}{da}\Gamma_{\sigma'_0}=0.$$

These are the differential equations of geodesics on the surface whose parameters are τ_0, σ_0.

Multiply (32) by τ_0' and (33) by σ_0' and add, remembering that

$$\bar{x}_{\tau_0}\tau_0'+\bar{x}_{\sigma_0}\sigma_0'=\bar{x}'.$$

The result is the single equation

$$\begin{aligned}&\bar{x}'\left[\bar{f}_x\mathbf{\Gamma}+\frac{d}{da}\left(\bar{f}_\tau\frac{\sin\bar{\tau}}{\cos\bar{\sigma}}+\bar{f}_\sigma\sin\bar{\sigma}\cos\bar{\tau}\right)\right]\\ &+\bar{y}'\left[\bar{f}_y\mathbf{\Gamma}-\frac{d}{da}\left(\bar{f}_\tau\frac{\cos\bar{\tau}}{\cos\bar{\sigma}}-\bar{f}_\sigma\sin\bar{\sigma}\sin\bar{\tau}\right)\right]+\bar{z}'\left[\bar{f}_z\mathbf{\Gamma}-\frac{d}{da}\bar{f}_\sigma\cos\bar{\sigma}\right]\\ &+\tau_0'\mathbf{\Gamma}_{\tau_0}+\sigma_0'\mathbf{\Gamma}_{\sigma_0}-\tau_0'\frac{d}{da}\bar{f}\mathbf{\Gamma}_{\tau'_0}-\sigma_0'\frac{d}{da}\bar{f}\mathbf{\Gamma}_{\sigma'_0}=0.\end{aligned}\tag{34}$$

Since $\mathbf{\Gamma}$ satisfies the Weierstrass homogeneity condition, the last four terms of (34) may be written

$$(1-\bar{f})\,(\tau_0'\mathbf{\Gamma}_{\tau_0}+\sigma_0'\mathbf{\Gamma}_{\sigma_0})-\mathbf{\Gamma}\frac{d}{da}\bar{f}=0.$$

Dividing (34) by $\mathbf{\Gamma}$, which is assumed to be different from zero, the equation becomes

$$\begin{aligned}&\cos\bar{\sigma}\cos\bar{\tau}\left[\bar{f}_x\mathbf{\Gamma}+\frac{d}{da}\left(\bar{f}_\tau\frac{\sin\bar{\tau}}{\cos\bar{\sigma}}+\bar{f}_\sigma\sin\bar{\sigma}\cos\bar{\tau}\right)\right]\\ &+\cos\bar{\sigma}\sin\bar{\tau}\left[\bar{f}_y\mathbf{\Gamma}-\frac{d}{da}\left(\bar{f}_\tau\frac{\cos\bar{\tau}}{\cos\bar{\sigma}}-\bar{f}_\sigma\sin\bar{\sigma}\sin\bar{\tau}\right)\right]\\ &+\sin\bar{\sigma}\left[\bar{f}_z\mathbf{\Gamma}-\frac{d}{da}\bar{f}_\sigma\cos\bar{\sigma}\right]+\frac{1-\bar{f}}{\mathbf{\Gamma}}\,(\tau_0'\mathbf{\Gamma}_{\tau_0}+\sigma_0'\mathbf{\Gamma}_{\sigma'_0})-\frac{d}{da}\bar{f}=0.\end{aligned}$$

If a is replaced by s, the Euclidean length of arc, $\mathbf{\Gamma}=1$, and the last equation reduces to

$$\begin{aligned}&\cos\bar{\sigma}\cos\bar{\tau}\left[\bar{f}_x+\frac{d}{ds}\left(\bar{f}_\tau\frac{\sin\bar{\tau}}{\cos\bar{\sigma}}+\bar{f}_\sigma\sin\bar{\sigma}\cos\bar{\tau}\right)\right]\\ &+\cos\bar{\sigma}\sin\bar{\tau}\left[\bar{f}_y-\frac{d}{ds}\left(\bar{f}_\tau\frac{\cos\bar{\tau}}{\cos\bar{\sigma}}-\bar{f}_\sigma\sin\bar{\sigma}\sin\bar{\tau}\right)\right]\\ &+\sin\bar{\sigma}\left[\bar{f}_z-\frac{d}{ds}\bar{f}_\sigma\cos\bar{\sigma}\right]-\frac{d}{ds}\bar{f}=0.\end{aligned}$$

§ 8. *A Generalization of the Definition of Solid Angle.*

The definition of the measure of a solid angle is the area cut out on a unit sphere by the cone which makes the solid angle (the vertex of the cone of course being at the center of the sphere), or what is the same thing, the ratio of the area cut out on any sphere to the square of the radius of that sphere. In the present section is given a generalized definition of solid angle, which for Euclidean geometry reduces to the ordinary definition.

If on the transversal surface considered in the preceding section there is a closed curve $\overline{C}$, the area $\overline{S}$ enclosed by $\overline{C}$ is

$$\overline{S}=\iint_S \sqrt{EG-F^2}\,d\tau_0 d\sigma_0,$$

where S is the image, in the τ_0, σ_0 plane of $\overline{S}$. A more general definition of of such area is obtained by setting

$$\overline{S}=\iint_S g(\tau_0,\ \sigma_0,\ \tau_0',\ \sigma_0')\sqrt{EG-F^2}\,d\tau_0 d\sigma_0.$$

Solid angle may then be defined by

$$\lim_{\rho=0}\frac{\overline{S}}{\rho^2}.$$

The values of

$$\lim_{\rho=0}\frac{E}{\rho^2},\quad \lim_{\rho=0}\frac{F}{\rho^2},\quad \lim_{\rho=0}\frac{G}{\rho^2}$$

have already been found; from these values can be obtained

$$\lim_{\rho=0}\frac{1}{\rho^4}(EG-F^2)=\frac{1}{f^8}(f^2\cos^2\sigma_0+f_\tau^2)(f^2+f_\sigma^2)-\frac{1}{f^8}f_\tau^2 f_\sigma^2,$$

from which it follows that

$$\lim_{\rho=0}\frac{\overline{S}}{\rho^2}=\iint_S\frac{g}{f^4}\sqrt{(f^2\cos^2\sigma_0+f_\tau^2)(f^2+f_\sigma^2)-f_\tau^2 f_\sigma^2}\,d\tau_0 d\sigma_0.$$

It is obvious that for Euclidean geometry, when $f=g=1$ and the transversal surface is a sphere, this last expression reduces to the usual definition of solid angle as given above.

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